Oscillatory integrals with variable Calderón-Zygmund kernel on vanishing generalized Morrey spaces

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Abstract

In this paper, the authors investigate the boundedness of the oscillatory singular integrals with variable Calderón-Zygmund kernel on generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM^{p,\varphi}(\mathbb{R}^n)$. When $1 < p < \infty$ and (φ_1, φ_2) satisfies some conditions, we show that the oscillatory singular integral operators T_λ and T_λ^* are bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$ and from $VM^{p,\varphi_1}(\mathbb{R}^n)$ to $VM^{p,\varphi_2}(\mathbb{R}^n)$. Meanwhile, the corresponding result for the oscillatory singular integrals with standard Calderón-Zygmund kernel are established.

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1 Introduction and main results

The classical Morrey spaces were originally introduced by Morrey in [18] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [3, 6, 7, 12, 18, 20, 22, 23, 24, 26]. Guliyev, Mizuhara and Nakai [10, 17, 19] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [11, 12, 27]). In [10, 12, 17, 19], the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi}$ was also studied, see also [1, 8, 13, 29].

Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any $f \in L^p_{\rm loc}(\mathbb{R}^n)$, we denote by $M^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey spaces, if

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L^p(B(x,r))} < \infty.$$

When $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, then $M^{p,\varphi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces. There are many papers discussed the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [12], the following condition was imposed on the pair (φ_1, φ_2):

$$\int_r^\infty \frac{\varphi_1(x,t)}{t} dt \le C\, \varphi_2(x,r), \qquad (1.1)$$

where C does not depend on x and t. Under the above condition, Guliyev obtained the boundedness of the singular integral operator T from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_1}(\mathbb{R}^n)$. Recently, in [1, 13], Guliyev et.

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introduced a weaker condition: If $1 < p < \infty$, for any $x \in \mathbb{R}^n$ and $t > 0$, there exits a constant $c > 0$, such that

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r). \tag{1.2}$$

If the pair (φ_1, φ_2) satisfies condition (1.1), then (φ_1, φ_2) satisfied condition (1.1). But the opposite is not true. We can see remark 4.7 in [13] for details.

Suppose that k is the standard Calderón-Zygmund kernel. That is, $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n$, and $\int_\Sigma k(x) d\sigma_x = 0$, where $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$. The oscillatory integral operator T_λ is defined by

$$T_\lambda f(x) = p.v. \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} k(x-y)\varphi(x,y) f(y) dy, \tag{1.3}$$

where $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n \times \mathbb{R}^n$ with compact supports, and Φ is a real-analytic function or a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp} \varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2 \Phi(x_0, y_0)/\partial x_{j_0} \partial y_{k_0}$ does not vanish up to infinite order. These operators have arisen in the study of singular integrals supported on lower dimensional varieties, and the singular Radon transform. In [21], Y. B. Pan proved that T_λ are uniformly in λ bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Let $k(x, y)$ be a variable Calderón-Zygmund kernel. That means, for a. e. $x \in \mathbb{R}^n, k(x, .)$ is a standard Calderón-Zygmund kernel and

$$\max_{|j| \le 2n, j \in \mathbb{N}_0^n} \left\| \frac{\partial^{|j|} k}{\partial y^j} \right\|_{L^\infty(\mathbb{R}^n \times \Sigma)} = A < \infty. \tag{1.4}$$

Define the oscillatory integral operator with variable Calderón-Zygmund kernel T_λ^* by

$$T_\lambda^* f(x) = p.v \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} k(x, x-y)\varphi(x,y) f(y) dy, \tag{1.5}$$

where λ, φ and Φ satisfy the same assumptions as those in the operator defined by (1.3).

S. Z. Lu and D. C. Yang etc. [16] investigated the L^p boundedness about this class of oscillatory integral operators. The boundedness of some operators on these spaces can be see ([1, 10, 12, 13, 17, 18, 19, 28, 29],). Recently, A. Eroglu [15] obtained the boundedness of a class of oscillatory integral with Calderón-Zygmund kernel and polynomial phase on generalized Morrey spaces.

The purpose of this paper is to generalize the results above to the case with real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ or analytic phase functions. Our main results in this paper are formulated as follows.

Theorem 1.1. Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp} \varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2 \Phi(x_0, y_0)/\partial x_{j_0} \partial x_{k_0}$ does not vanish up to infinite order. Assume k is a standard Calderón-Zygmund kernel and T_λ is defined as in (1.3). Then for any $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega_p$ satisfies the condition (1.2), the operator T_λ is bounded from M^{p,φ_1} to M^{p,φ_2}.

Theorem 1.2. Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp} \varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2 \Phi(x_0, y_0)/\partial x_{j_0} \partial x_{k_0}$ does not vanish up to infinite order. Assume k is a variable Calderón-Zygmund kernel and T^*_λ is defined as in (1.5). Then for any $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega_p$ satisfies the condition (1.2), the operator T^*_λ is bounded from M^{p,φ_1} to M^{p,φ_2}.

Theorem 1.3. Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume k is a standard Calderón-Zygmund kernel and T_λ is defined as in (1.3). Then for any $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega_{p,1}$ satisfies the conditions

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty \tag{1.6}$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t} \le C_0\varphi_2(x,r), \tag{1.7}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$, the operator T_λ is bounded from VM^{p,φ_1} to VM^{p,φ_2}.

Theorem 1.4. Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \le j_0, k_0 \le n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume k is a standard Calderón-Zygmund kernel and T^*_λ is defined as in (1.5). Then for any $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega_{p,1}$ satisfies the conditions (1.6) and (1.7), the operator T^*_λ is bounded from VM^{p,φ_1} to VM^{p,φ_2}.

2 Notations and preliminary Lemmas

Let $B = B(x_0, r)$ be the ball with the center x_0 and radius r. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B.

Lemma 2.1. [9] Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p < \infty$.

(i) If

$$\sup_{t<r<\infty} r^{-n/p}\,\varphi(x,r)^{-1} = \infty \quad \text{for some } t > 0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.8}$$

then $M^{p,\varphi}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.9}$$

then $M^{p,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.2. We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| r^{-n/p}\,\varphi(x,r)^{-1} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}(f;x,r) := r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}_{W,p,\varphi}(f;x,r) := r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))},$$

where $WL^p(B(x,r))$ denotes the weak L^p-space of measurable functions f for which

$$\|f\|_{WL^p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL^p(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\,|f(y)|>t\}} dy\right)^{\frac{1}{p}}.$$

Definition 2.3. The vanishing generalized Morrey space $VM^{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M^{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi}(f;x,r) = 0. \tag{2.10}$$

The vanishing weak generalized Morrey space $VWM^{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM^{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{W,p,\varphi}(f;x,r) = 0.$$

The vanishing spaces $VM^{p,\varphi}(\mathbb{R}^n)$ and $VWM^{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{p,\varphi}} \equiv \|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{p,\varphi}(f;x,r),$$
$$\|f\|_{VWM^{p,\varphi}} \equiv \|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{W,p,\varphi}(f;x,r),$$

respectively.

Remark 2.4. We denote by $\Omega_{p,1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta} \varphi(x,r) > 0, \text{ for some } \delta > 0, \tag{2.11}$$

and

$$\lim_{r\to 0}\frac{r^{n/p}}{\varphi(x,r)} = 0. \tag{2.12}$$

For the non-triviality of the space $VM^{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,1}$.

The vanishing generalized Morrey space $VM^{p,\varphi}(\mathbb{R}^n)$ were studied in [2]. In the case $\varphi(x,r) = r^{(\lambda-n)/p}$, $VM^{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ introduced by Vitanza in [31], where applications to PDE were considered. We refer to [5, 14, 22, 25] for some properties of vanishing generalized Morrey spaces.

Our argument based heavily on the following results.

Lemma 2.5. [16] Assume T_λ is defined as in (1.3). Then for any $1 < p < \infty$, we have

$$\|T_\lambda f\|_{L^p} \leq C(n,p,\Phi,\varphi,C_p)\, B\, \|f\|_{L^p},$$

where $C(n,p,\Phi,\varphi,C_p)$ is independent of λ, k and f, and $B = \|k\|_{C^1(\sum)}$.

Lemma 2.6. [16] Assume T^*_λ is defined as in (1.5). Then for any $1 < p < \infty$, we have

$$\|T^*_\lambda f\|_{L^p} \leq C(n,p,\Phi,\varphi,C_p)\, A\, \|f\|_{L^p},$$

where $C(n,p,\Phi,\varphi,C_p)$ is independent of λ, k and f. A is defined in (1.4).

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \qquad f \in L_{loc}(\mathbb{R}^n).$$

Theorem 2.7. [1, 13] Let $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition (1.2). Then the maximal operator M and the singular integral operator T are bounded from M^{p,φ_1} to M^{p,φ_2} for $p > 1$ and from M^{1,φ_1} to WM^{1,φ_2}.

A distribution kernel k is called a standard Calderòn-Zygmund kernel (SCZK) if it satisfies the following hypotheses:

$$|k(x,y)| \leq \frac{C}{|x-y|^n}, \forall x \neq y,$$

$$|\nabla_x k(x,y)| + |\nabla_y k(x,y)| \leq \frac{C}{|x-y|^{n+1}}, \forall x \neq y.$$

The corresponding Calderòn-Zygmund integral operator S and oscillatory integral operator R are defined by

$$Sf(x) = p.v. \int_{\mathbb{R}^n} k(x,y) f(y) dy$$

and

$$Rf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} k(x,y) f(y) dy,$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem A [15] Let $1 < p < \infty$, and (φ_1, φ_2) satisfies the condition (1.1). If S is of type (L^2, L^2), then for any real polynomial $P(x,y)$, there exists a constant $C > 0$ such that

$$\|Rf\|_{M^{p,\varphi_2}} \leq C \|f\|_{M^{p,\varphi_1}}.$$

Lemma 2.8. [30] Denote by $\mathcal{H}_m$ the spaces of spherical harmonic functions of degree m. Then

(a) $L^2(\sum) = \oplus_{m=0}^{\infty} \mathcal{H}_m$, *and* $g_m = dim \mathcal{H}_m \leq C(n) m^{n-2}$ for any $m \in \mathbb{N}$,

(b) for any $m = 0, 1, 2, ...$, there exists an orthogonal system $\{Y_{jm}\}_{j=1}^{g_m}$ of $\mathcal{H}_m$ such that $\|Y_{jm}\|_{L^\infty(\sum)} \leq C(n) m^{n/2-1}, Y_{jm} = (-m)^{-n}(m+n-2)^{-n} \Lambda^n Y_{jm}, j = 1, ..., g_m$, and Λ is the Beltrami-Laplace operator on $\sum$.

In the following the letter C will denote a constant which may vary at each occurrence.

3 Proof of Theorems 1.1 and 1.2

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$Hg(t) := \frac{1}{t}\int_0^t g(r)dr, \ 0 < t < \infty.$$

Theorem B. [4] *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname*{ess\,inf}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

The following lemma is valid.

Lemma 3.1. Let $1 < p < \infty$ and T_λ is defined as in (1.3). Then the inequality

$$\|T_\lambda f\|_{L^p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt \tag{3.13}$$

holds for any ball $B(x_0,r)$ and for all $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)^\complement}(y)$$

and have

$$\|T_\lambda f\|_{L^p(B)} \le \|T_\lambda f_1\|_{L^p(B)} + \|T_\lambda f_2\|_{L^p(B)}.$$

It is known that (see Lemma 2.5) the operator T_λ is bounded on $L^p(\mathbb{R}^n)$. Since $f_1 \in L^p(\mathbb{R}^n)$, $T_\lambda f_1 \in L^p(\mathbb{R}^n)$ and boundedness of T_λ in $L^p(\mathbb{R}^n)$ (see [16]) it follows that

$$\|T_\lambda f_1\|_{L^p(B)} \le \|T_\lambda f_1\|_{L^p(\mathbb{R}^n)} \le C\|f_1\|_{L^p(\mathbb{R}^n)} = C\|f\|_{L^p(2B)},$$

where the constant $C > 0$ is independent of f.

We now estimate $T_\lambda f_2$. We can write

$$\Big|T_\lambda f_2(x)\Big| = \Big|\int_{(2B)^\complement} e^{i\lambda\Phi(x,y)} k(x-y)\varphi(x,y)f(y)dy\Big|.$$

Now by an argument similar to the proof of Lemma 6 in [16], we choose $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_1(x) \equiv 1$ when $|x| \le 1$, and $\varphi_1(x) \equiv 0$ when $|x| > 2$. Let $\varphi_2 = 1 - \varphi_1$ and $N \in \mathbb{N}$ which is large enough and will be determined later. Write

$$k(x) = k^1_\lambda(x) + k^2_\lambda(x),$$

where

$$k_\lambda^j = k(x)\varphi_j(\lambda^{1/N}x),\ \ j=1,2.$$

Then

$$T_\lambda f_2(x) = p.v. \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_\lambda^1(x-y)\varphi(x,y)f(y)dy$$
$$+ p.v. \int_{(2B)^c} e^{i\lambda\Phi(x,y)} k_\lambda^2(x-y)\varphi(x,y)f(y)dy := T_\lambda^1 f_2(x) + T_\lambda^2 f_2(x).$$

Let us first estimate $T_\lambda^1 f_2(x)$. To do so, using Taylor's expansion and the compactness of $\operatorname{supp}\varphi$, we write

$$\Phi(x,y) = \Phi(x,x) + P(x,y) + r_N(x,y)$$

for $(x.y) \in \operatorname{supp}\varphi$, where $P(x,y)$ is a polynomial with deg $P < N$ and $|r_N(x,y)| \le C|x-y|^N$ with C in dependent of x and y. Define

$$Rf(x) = p.v. \int_{(2B)^c} e^{i\lambda P(x,y)} k_\lambda^1(x-y)\varphi(x,y)f(y)dy.$$

Therefore

$$\begin{aligned}
&e^{-i\lambda\Phi(x,x)} T_\lambda^1 f_2(x) - Rf(x)\\
&= \int_{|x-y|\le 2\lambda^{-1/N}} e^{i\lambda P(x,y)} \big(e^{i\lambda r_N(x,y)} - 1]\big) k_\lambda^1(x-y)\varphi(x,y)f(y)dy\\
&= \sum_{j=0}^{\infty} \int_{2^{-j}\lambda^{-1/N} < |x-y| \le 2^{-j+1}\lambda^{-1/N}} e^{i\lambda P(x,y)} \big(e^{i\lambda r_N(x,y)} - 1\big) k_\lambda^1(x-y)\varphi(x,y)f(y)dy\\
&\equiv \sum_{j=0}^{\infty} T_{\lambda j}^1 f_2(x).
\end{aligned}$$

On $T_{\lambda j}^1 f_2(x)$, by the properties of r_N and k, we have

$$|T_{\lambda j}^1 f_2(x)| \le C2^{-jN} Mf(x).$$

So we have

$$|T_\lambda^1 f_2(x)| \le C\sum_{j=0}^{\infty} 2^{-jN} Mf(x) + C|Rf(x)| \le CMf(x) + C|Rf(x)|.$$

By Theorem 4.1 in [12] and Lemma 3.1 in [15], we have

$$\begin{aligned}
\|T_\lambda^1 f_2\|_{L^p(B(x_0,r))} &\lesssim \|Mf\|_{L^p(B(x_0,r))} + \|Rf\|_{L^p(B(x_0,r))}\\
&\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt.
\end{aligned}$$

Now, let us turn to estimate $T_\lambda^2 f_2(x)$. We consider the following two cases.
Case 1. $\lambda \le 1$. Similar to that estimate of T_λ^2 in Lemma 6 in [16], we have

$$|T_\lambda^2 f_2(x)| \le CMf(x),$$

where the constant $C > 0$ is independent of f. By Theorem 4.1 in [12] we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r))} \le C\|f\|_{L^p(B(x_0,r))}.$$

Case 2. $\lambda > 1$. We choose $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\varphi_0 \subseteq \{x \in \mathbb{R}^n : 1 < |x| \le 2\},$$

and

$$\varphi_2(x) = \sum_{j=0}^{\infty} \varphi_0(2^{-j}x).$$

Let

$$k_{\lambda,j}^2(x) = k(x)\varphi_0(2^{-j}\lambda^{1/N}x).$$

Then

$$\begin{aligned} T_\lambda^2 f_2(x) &= \int_{(2B)^\complement} e^{i\lambda\Phi(x,y)} k_\lambda^2(x-y)\varphi(x,y)f(y)dy \\ &= \sum_{j=0}^{\infty} \int_{(2B)^\complement} e^{i\lambda\Phi(x,y)} k_{\lambda,j}^2(x-y)\varphi(x,y)f(y)dy \\ &\equiv \sum_{j=0}^{\infty} T_{\lambda,j}^2 f_2(x). \end{aligned}$$

For $T_{\lambda,j}^2$, by the definition of it, we can get

$$|T_\lambda^2 f_2(x)| \le C \int_{2^j\lambda^{-1/N} < |x-y| \le 2^{j \cdot 1}\lambda^{-1/N}} \frac{1}{|x-y|^n}|f(y)|dy \le CMf(x). \tag{3.14}$$

The inequality (3.14) also can be see in [16], we omit the detail here.
By Theorem 4.1 in [12], we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r))} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt.$$

Therefore

$$\begin{aligned} \|T_\lambda f_2\|_{L^p(B(x_0,r))} &\le \|T_\lambda^1 f_2\|_{L^p(B(x_0,r))} + \|T_\lambda^2 f_2\|_{L^p(B(x_0,r))} \\ &\le Cr^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt. \end{aligned}$$

This finishes the proof of Lemma 3.1. Q.E.D.

Proof of Theorem 1.1

By Lemma 3.1 and Theorem B we get

$$\begin{aligned}\|T_\lambda f\|_{M^{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L^p(B(x,t))} t^{-1-\frac{n}{p}} dt \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L^p(B(x,t^{-\frac{p}{n}}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{p}{n}})^{-1} \int_0^{r} \|f\|_{L^p(B(x,t^{-\frac{p}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{p}{n}})^{-1}\, r\, \|f\|_{L^p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{M^{p,\varphi_1}}\end{aligned}$$

This finishes the proof of Theorem 1.1.

The following lemma is valid.

Lemma 3.2. Let $1 < p < \infty$ and T_λ^* is defined as in (1.5). Then the inequality

$$\|T_\lambda^* f\|_{L^p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)^\complement}(y)$$

and have

$$\|T_\lambda^* f\|_{L^p(B)} \le \|T_\lambda^* f_1\|_{L^p(B)} + \|T_\lambda^* f_2\|_{L^p(B)}.$$

It is known that (see Lemma 2.5) the operator T_λ^* is bounded on $L^p(\mathbb{R}^n)$. Since $f_1 \in L^p(\mathbb{R}^n)$, $T_\lambda^* f_1 \in L^p(\mathbb{R}^n)$ and boundedness of T_λ^* in $L^p(\mathbb{R}^n)$ (see [16]) it follows that

$$\|T_\lambda^* f_1\|_{L^p(B)} \le \|T_\lambda^* f_1\|_{L^p(\mathbb{R}^n)} \le C\|f_1\|_{L^p(\mathbb{R}^n)} = C\|f\|_{L^p(2B)},$$

where the constant $C > 0$ is independent of f.

We now estimate $T_\lambda^* f_2$. For each $m \in \mathbb{N}$ and $j = 1, \ldots, g_m$, we get

$$a_{jm}(x) = \int_\Sigma \Omega(x,z) Y_{jm}(z) d\sigma_z,$$

where $\Omega(x,z) = |z|^n k(x,z)$. Then for $a.e. x \in \mathbb{R}^n$,

$$\Omega(x,z) = \sum_{m=1}^\infty \sum_{j=1}^{g_m} a_{jm}(x) Y_{jm}(z'), \tag{3.15}$$

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \backslash \{0\}$. By Lemma 2.8, we have that for any $x \in \mathbb{R}^n$,

$$\begin{aligned}|a_{jm}(x)| &= m^{-n}(m+n-2)^{-n} \left| \int_{\sum} \Omega(x,z) \Lambda^n Y_{jm}(z) d\sigma_z \right| \\ &= m^{-n}(m+n-2)^{-n} \left| \int_{\sum} \Lambda^n \Omega(x,z) Y_{jm}(z) d\sigma_z \right| \\ &\le C(n) A m^{-2n}. \end{aligned} \tag{3.16}$$

By Lemma 2.8 again, we can verify that for any $\varepsilon > 0, N \in \mathbb{N}$, and a.e. $x \in \mathbb{R}^n$, if $|y - x| \ge \varepsilon$, then

$$\left| \sum_{m=1}^{N} \sum_{j=1}^{g_m} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) \right| \le C(\varepsilon)\, A\, |f_2(y)|. \tag{3.17}$$

Therefore, from (3.15), (3.17) and the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned} T_\lambda^* f_2(x) &= \lim_{\varepsilon \to 0} \int_{|x-y| \ge \varepsilon} e^{i\lambda\Phi(x,y)} k(x, x-y) \varphi(x,y) f_2(y) dy \\ &= \lim_{\varepsilon \to 0} \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} \int_{|x-y| \ge \varepsilon} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) dy \\ &= \lim_{\varepsilon \to 0} \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} a_{jm}(x) \int_{|x-y| \ge \varepsilon} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) dy. \end{aligned}$$

We write

$$R_{jm} f_2(x) = \int_{|x-y| \ge \varepsilon} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) dy.$$

It is easy to see that $R_{jm} f_2(x)$ is the oscillatory integral operator defined by (1.3). By Theorem 1.1 we have R_{jm} bounded from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$. Therefore, by (3.16) and the above discussion we have

$$\|T_\lambda^* f_2\|_{L^p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-1-\frac{n}{p}} dt.$$

This finishes the Lemma 3.2. Q.E.D.

Proof of Theorem 1.2.

By Lemma 3.2 and Theorem B we get

$$\begin{aligned}
\|T^*_\lambda f\|_{M^{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L^p(B(x,t))} t^{-1-\frac{n}{p}} dt \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L^p(B(x,t^{-\frac{p}{n}}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{p}{n}})^{-1} \int_0^{r} \|f\|_{L^p(B(x,t^{-\frac{p}{n}}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{p}{n}})^{-1}\, r\, \|f\|_{L^p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{M^{p,\varphi_1}}
\end{aligned}$$

This finishes the proof of Theorem 1.2.

4 Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3.

The statement is derived from the estimate (3.13). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1.1. So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi_1}(f;x,r) = 0 \quad \Rightarrow \quad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi_2}(T_\lambda f;x,r) = 0. \tag{4.18}$$

To show that $\sup_{x\in\mathbb{R}^n} \varphi_2(x,r)^{-1} r^{-n/p} \|T_\lambda f\|_{L_p(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (3.13):

$$\varphi_2(x,r)^{-1} r^{-n/p} \|T_\lambda f\|_{L_p(B(x,r))} \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{4.19}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \varphi_1(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.7) and (4.19). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.11) we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0}\ \frac{1}{\varphi_1(x,r)}\ \|f\|_{VM^{p,\varphi_1}},$$

where c_{σ_0} is the constant from (2.10). Then, by (2.11) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{p,\varphi_1}}},$$

which completes the proof of (4.18).

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3.

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